ISRAEL JOURNAL OF MATHEMATICS, Vol. 54, No. 2, 1986

EXTENDING IDEALS

BY
JURIS STEPRĀNS AND W. STEPHEN WATSON
Department of Mathematics, York University, Downsview, Ontario M3J 1P3, Canada

ABSTRACT
In this paper the following two questions are studied: (1) When does an ideal I have the property that whenever A is a family of κ sets there is a σ-ideal which extends I and measures each element of A? (2) When does an ideal I have the property that when A is a family of κ sets there is a σ-ideal which extends I and measures at least λ elements of A?

The above questions are surprisingly ubiquitous in combinatorial set theory: polarized and other partition relations, saturation of ideals, Silver's principles, character of ultrafilters, HFDs, independent sets are all intrinsically related to extendibility. The methods used in solving these questions are just as diverse: the core model, preservation of linguistic and other aspects of weak compactness under various forcing extensions, ZFC combinatorics, huge cardinals, Sacks, Mathias and other reals and more.

We have restricted this paper in several ways: We have examined only "first order" and "second order" extendibility. Perhaps "third order" extendibility is even more interesting. We have also emphasized the relative consistency of extendibility with possible cardinal arithmetic. Perhaps there are interesting implications betweeen extendibilities and other axioms. The topics we have selected are only a basic framework for examining extendibility properties.

Some definitions will help to simplify the rest of the discussion. If I is an ideal on X then *I measures A* if and only if A is a subset of X and either $A \in I$ or $X \setminus A \in I$. The ideal I is *κ-extendible* if and only if whenever $A \in [\mathcal{P}(X)]^\kappa$ there is a σ-ideal (i.e. a countably complete ideal) extending I which measures each element of A. The ideal I is *(κ, λ)-extendible* if and only if whenever $A \in [\mathcal{P}(X)]^\kappa$ there is $\bar{A} \in [A]^\lambda$ and there is a σ-ideal extending I which measures each element of $\bar{A}$.

Received January 13, 1983 and in revised form June 17, 1985

The questions examined in this paper can now be posed as follows:
(1) When is an ideal κ-extendible?
(2) When is an ideal (κ, λ)-extendible?

Part One: When is an ideal κ-extendible?

A definition simplifies the statement of the answer: I is *κ-completable* if there is a κ-complete ideal on X which contains I. We shall show that the extendibility of an ideal is, in general, determined by its completability.

First, we give a sufficient condition for κ-extendibility.

LEMMA 1. *If I is $(\kappa^\omega)^+$-completable then I is κ-extendible.*

PROOF. Assume, without loss of generality, that I is $(\kappa^\omega)^+$-complete. Let $\{A_\mu : \mu < \kappa\}$ be a family of subsets of X. For each $\mu < \kappa$, let $A^0_\mu = A_\mu$ and $A^1_\mu = X - A_\mu$. Let

$$N = \bigcup\{\bigcap\{A^{f(\mu)}_\mu : \mu \in L\} : L \in [\kappa]^{\aleph_0} \text{ and } f : L \to 2 \text{ and } \bigcap\{A^{f(\mu)}_\mu : \mu \in L\} \in I\}.$$

There are at most κ^ω possible countable subsets L of κ, and for each L at most 2^ω possible $f : L \to 2$. Thus N is the union of a family of κ^ω elements of I. Since I is $(\kappa^\omega)^+$-complete, N is in the ideal and thus $X - N$ is nonempty. Choose $\alpha \in X - N$ and define $g : \kappa \to 2$ as follows: $g(\mu) = 0$ if and only if $\alpha \notin A_\mu$. Note that $\alpha \notin A^{g(\mu)}_\mu$ for each $\mu < \kappa$.

It will now be shown that the countable completion of $I \cup \{A^{g(\mu)}_\mu : \mu < \kappa\}$ exists. Since I is countably complete it suffices to show that $B \cup (\bigcup\{A^{g(\mu)}_\mu : \mu \in L\}) \neq X$ whenever $L \in [\kappa]^{\aleph_0}$ and $B \in I$. Suppose, otherwise, that $X - \bigcup\{A^{g(\mu)}_\mu : \mu \in L\} \in I$. If $f : L \to 2$ is defined by $f(\mu) = g(\mu) + 1 (\text{mod } 2)$ then $\bigcap\{A^{f(\mu)}_\mu : \mu \in L\} \in I$ and so $\bigcap\{A^{f(\mu)}_\mu : \mu \in L\} \subset N$. This implies that $\alpha \in N$ which contradicts the choice of α. The lemma is proved.

Second, we show that the sufficient condition of Lemma 1 is necessary for "small" cardinals.

The definition of a Ξ-cardinal, a pathological large cardinal, facilitates the description of the cardinals for which the sufficient condition of Lemma 1 may not be a necessary condition.

κ is a *Ξ-cardinal* whenever:

(1) κ is a regular limit cardinal greater than the continuum,

(2) κ is a weakly compact cardinal in L,

(3) κ cannot be obtained by nontrivial cardinal exponentiation (that is, there do not exist $\alpha, \beta < \kappa$ such that $\kappa = \alpha^\beta$),

(4) κ is not inaccessible.

A Ξ-cardinal has the same consistency strength as the existence of a weakly compact cardinal.

LEMMA 2. *If I is κ-extendible then I is $(\kappa^{\omega})^{+}$-completable unless κ is greater than or equal to either a weakly compact cardinal or a Ξ-cardinal.*

PROOF. Suppose I fails to be $(\kappa^{\omega})^{+}$-completable. We construct a tree of height ω. Each node consists of an ordered pair. The first coordinate is an indexed partition of X. The cardinality of the first coordinate therefore equals the cardinality of the index set. This allows the partitions to contain the empty set. The second coordinate is a family of at most κ subsets of X. The first coordinate of the unique node at level 0 is a partition of X into $\{A_f : f \in {}^{\omega}\kappa\}$ where each A_f is in I. The second coordinate of the node at level 0 is $\{\bigcup\{A_f : f(n) = \mu\} : n \in \omega, \mu \in \kappa\}$, a family of at most κ subsets of X. There are countably many nodes at level 1. The first coordinate of the nth node at level 1 is $\{\bigcup\{A_f : f(n) = \mu\} : \mu < \kappa\}$, a partition of X. The tree is defined by induction. If the first coordinate of a node has been defined, we define its second coordinate and the first coordinate of its successors (if there are any). The induction step at a node depends on the cardinality ν of the first coordinate of the node.

The induction begins at level 1 and the first coordinate of each node, except the node at level 0, has cardinality at most κ.

There are five possibilities for $\nu \leqq \kappa$:

(1) ν is a successor cardinal,

(2) ν is a singular cardinal,

(3) there are $\alpha, \beta < \nu$ such that $\nu \leqq \alpha^{\beta}$ and either $\beta = \omega$ or $\alpha^{\beta} \leqq \kappa$,

(4) there is a ν-Aronszajn tree,

(5) $\nu = \omega$.

A $\nu \leqq \kappa$ may have more than one possibility. It does not matter which set of instructions are applied.

For the first four cases, a set-theoretic object is needed:

(1) an Ulam matrix,

(2) a singular chain,

(3) a Cantor tree,

(4) an Aronszajn tree.

Let the first coordinate of a node be $\{B_\alpha : \alpha < \nu\}$, a partition of X.

Case 1. Let $\nu = \rho^{+}$. Construct an Ulam matrix $\{u_{\sigma\tau} : \sigma < \upsilon, \tau < \rho\}$ of subsets of ν such that $\{u_{\sigma\tau} : \sigma < \nu\}$ is a disjoint family for each $\tau < \rho$ and $\{u_{\sigma\tau} : \tau < \rho\}$ is a partition of $\nu - \sigma$ for each $\sigma < \nu$. Let the second coordinate of the node be

$\{\bigcup\{B_\alpha : \alpha \in u_{\sigma\tau}\} : \sigma < \nu, \tau < \rho\}$, a family of cardinality $\nu \leqq \kappa$. The node has ν successors. The σth successor, where $\sigma < \nu$, has first coordinate

$$\{\bigcup\{B_\alpha : \alpha \in u_{\sigma\tau}\} : \tau < \rho\} \cup \{B_\alpha : \alpha < \sigma\},$$

a partition of X of cardinality $\rho + |\sigma| = \rho < \nu$.

Case 2. Let $\mathrm{cf}(\nu) = \rho$. Construct a partition $\{u_\sigma : \sigma < \rho\}$ of ν such that $|u_\sigma| < \nu$ for each $\sigma < \rho$. Let the second coordinate of the node be $\{\bigcup\{B_\alpha : \alpha \in u_\sigma\} : \sigma < \rho\}$, a family of cardinality $\rho \leqq \nu \leqq \kappa$. The node has a successor for each $\sigma \leqq \rho$. The σth successor, where $\sigma < \rho$, has first coordinate

$$\{B_\alpha : \alpha \in u_\sigma\} \cup \{\bigcup\{B_\alpha : \alpha \notin u_\sigma\}\},$$

a partition of X of cardinality $|u_\sigma| < \nu$. The ρth successor has first coordinate $\{\bigcup\{B_\alpha : \alpha \in u_\sigma\} : \sigma < \rho\}$ a partition of X of cardinality $\rho < \nu$.

Case 3. Let α and β be less than ν such that $\nu \leqq \alpha^\beta$ and let $\pi : \nu \to \beta_\alpha$ be an injection. Let the second coordinate of the node be

$$\{\bigcup\{B_\xi : \pi(\xi)(\sigma) = \tau\} : \sigma < \beta, \tau < \alpha\},$$

a family of cardinality $\alpha + \beta < \nu \leqq \kappa$. If β is uncountable then the node has a successor for each $f : \beta \to \alpha$. There are α^β such successors and $\alpha^\beta \leqq \kappa$. If $f \in {}^\beta\alpha$ then the successor corresponding to f has first coordinate

$$\{\bigcup\{B_\xi : \pi(\xi)\restriction\sigma = f\restriction\sigma \text{ and } \pi(\xi)(\sigma) \neq f(\sigma)\} : \sigma < \beta\} \cup \{B_\xi : \pi(\xi) = f\}.$$

The node also has a successor for each $\sigma < \beta$. The σth successor, where $\sigma < \beta$, has first coordinate $\{\bigcup\{B_\xi : \pi(\xi)(\sigma) = \tau\} : \tau < \alpha\}$.

Case 4. Let $(T, \lhd)$ be a tree of height ν all of whose levels and branches have cardinality less than ν and let $\pi : \nu \to T$ be a bijection. Let the second coordinate of the node be $\{\bigcup\{B_\alpha : t \lhd \pi(\alpha)\} : t \in T\}$. The node has ν successors. The βth successor, for $\beta < \nu$, has first coordinate

$$\{\bigcup\{B_\alpha : t \lhd \pi(\alpha)\} : \mathrm{level}(t) = \beta\} \cup \{B_\alpha : \mathrm{level}(\pi(\alpha)) \leqq \beta\},$$

a partition of X of cardinality less than ν.

Case 5. Let the second coordinate of the node be the empty set. The node has no successors.

The cardinality of the first coordinate decreases as one moves up a branch in the tree and so there are no infinite branches. The tree is at most κ-branching and so it has most κ nodes. The cardinality of each second coordinate is at most

κ. Let U be the union of the second coordinates of nodes in the tree. U is a family (of cardinality at most κ) of subsets of X. By way of contradiction, suppose J is a σ-ideal which contains I and measures each element of U.

To facilitate the exposition, let a *small* subset of X be a subset of X which is in J.

We construct a branch of the tree by induction on level. This branch has the property that each node of the branch has a first coordinate which is a partition of X into small subsets of X.

The node at level 0 has this property. It will be shown that each node with a successor has a successor with this property.

Case 1. For each $\sigma < \nu$ and $\tau < \rho$ the ideal J measures $\bigcup\{B_\alpha : \alpha \in u_{\sigma\tau}\}$. We show that there exists $\sigma_0 < \nu$ such that $\{\bigcup\{B_\alpha : \alpha \in u_{\sigma_0\tau}\} : \tau < \rho\}$ is a family of small sets. Otherwise, for each $\sigma < \nu$, there is a $\tau(\sigma) < \rho$ such that $\bigcup\{B_\alpha : \alpha \in u_{\sigma\tau(\sigma)}\} \notin J$. There must be $\sigma_1 < \nu$ and $\sigma_2 < \nu$ such that $\tau(\sigma_1) = \tau(\sigma_2)$. But $u_{\sigma_1\tau(\sigma_1)}$ is disjoint from $u_{\sigma_2\tau(\sigma_2)}$ and so $\bigcup\{B_\alpha : \alpha \in u_{\sigma_1\tau(\sigma_1)}\}$ and $\bigcup\{B_\alpha : \alpha \in u_{\sigma_2\tau(\sigma_2)}\}$ are disjoint elements with small complement which is a contradiction. Now the σ_0th successor has the desired property because its first coordinate is

$$\{\bigcup\{B_\alpha : \alpha \in u_{\sigma_0\tau}\} : \tau < \rho\} \cup \{B_\alpha : \alpha < \sigma_0\}.$$

But $\{B_\alpha : \alpha < \sigma_0\} \subset \{B_\alpha : \alpha < \nu\}$ and $\{B_\alpha : \alpha \in \nu\}$ is a partition of X into small sets by the induction hypothesis. Thus each element of the first coordinate of the σ_0th successor is small as required.

Case 2. For each $\sigma < \rho$ the ideal J measures $\bigcup\{B_\alpha : \alpha \in u_\sigma\}$. Suppose that there exists a $\sigma < \rho$ such that $\bigcup\{B_\alpha : \alpha \in u_\sigma\} \notin J$. To see that the σth successor has the property note that the first coordinate of the σth successor is

$$\{B_\alpha : \alpha \in u_\sigma\} \cup \{\bigcup\{B_\alpha : \alpha \notin u_\sigma\}\}.$$

However $\{B_\alpha : \alpha \in u_\sigma\}$ is contained in $\{B_\alpha : \alpha < \nu\}$ which is a partition of X into small sets, by the induction hypothesis.

Since $\bigcup\{B_\alpha : \alpha \notin u_\sigma\} = X - \bigcup\{B_\alpha : \alpha \in u_\sigma\}$ and the set on the right is small by the hypothesis on J it follows that each element of the first coordinate of the σth successor is small. If there does not exist $\sigma < \rho$ such that $\bigcup\{B_\alpha : \alpha \in u_\sigma\} \notin J$, then, for each $\sigma < \rho$, $\bigcup\{B_\alpha : \alpha \in u_\sigma\}$ is small. The first coordinate of the ρth successor consists of these sets and so each element of the first coordinate of the ρth successor is small.

Case 3. For each $\sigma < \beta$, the ideal J measures $\bigcup\{B_\xi : \pi(\xi)(\sigma) = \tau\}$. Define

$f:\beta\to\alpha$ as follows: $f(\sigma)=\tau$ if and only if $\bigcup\{B_\xi:\pi(\xi)(\sigma)=\tau\}\notin J$ for each $\sigma<\beta$. This definition is possible unless there is $\sigma<\beta$ such that $\bigcup\{B_\xi:\pi(\xi)(\sigma)=\tau\}$ is small for each $\tau<\alpha$ and so each element of the first coordinate of the σth successor is small as required. If β is uncountable, then we shall show that the successor has the property. The first coordinate of the fth successor is

$$\{\bigcup\{B_\xi:\pi(\xi)\restriction\sigma=f\restriction\sigma \text{ and } \pi(\xi)(\sigma)\neq f(\sigma)\}:\sigma<\beta\}\cup\bigcup\{B_\xi:\pi(\xi)=f\}.$$

Let $\sigma<\beta$. Then $\bigcup\{B_\xi:\pi(\xi)\restriction\sigma=f\restriction\sigma$ and $(\pi)(\xi)\sigma\neq f(\sigma)\}$ is contained in $\bigcup\{B_\xi:\pi(\xi)(\sigma)\neq f(\sigma)\}$. If $f(\sigma)=\tau$ then $\bigcup\{B_\xi:\pi(\xi)(\sigma)=\tau\}\notin J$ and hence $\bigcup\{B_\xi:\pi(\xi)(\sigma)\neq\tau\}$ is small. Also $B_{\pi^{-1}(f)}$ belongs to $\{B_\xi:\xi<\nu\}$ which is a partition of X into small sets by the induction hypothesis. Hence each element of the first coordinate of the fth successor is small as required. If β is countable, then $\{B_\xi:\pi(\xi)=f\}\cup\{\bigcup\{B_\xi:\pi(\xi)(\sigma)\neq f(\sigma)\}:\sigma<\beta\}$ is a countable partition of X into small sets and this is a contradiction.

Case 4. Now $\{B_\alpha:t\lhd\pi(\alpha)\}$ is measured by J for each $t\in T$. Suppose that $\{\bigcup\{B_\alpha:t\lhd\pi(\alpha)\}:\text{level}(t)=\beta\}$ consists of small sets for some $\beta<\nu$. To see that the βth successor has the desired property, note that βth successor has first coordinate

$$\{\bigcup\{B_\alpha:t\lhd\pi(\alpha)\}:\text{level}(t)=\beta\}\cup\{B_\alpha:\text{level}(\pi(\alpha))\leqq\beta\}.$$

But $\{B_\alpha:\text{level}(\pi(\alpha))\leqq\beta\}\subset\{B_\alpha:\alpha<\nu\}$, and $\{B_\alpha:\alpha\in\nu\}$ is a partition of X into small sets by the induction hypothesis, If, then, for each $\beta<\nu$, there is a t_β at level β such that

$$\bigcup\{B_\alpha:t_\beta\lhd\pi(\alpha)\}\notin J,\qquad \text{for each } \beta<\nu,$$

$\{\bigcup\{B_\alpha:t\lhd\pi(\alpha)\}:\text{level}(t)=\beta\}$ does not consist of small sets.

It will be shown if $\beta_1<\beta_2<\nu$ then $t_{\beta_1}\lhd t_{\beta_2}$. Suppose not. Then t_{β_2} has some predecessor t^* at level β_1. Clearly $\bigcup\{B_\alpha:t^*\lhd\pi(\alpha)\}\supset\bigcup\{B_\alpha:t_{\beta_2}\lhd\pi(\alpha)\}\notin J$. If t^* and t_{β_1} are distinct, then $\bigcup\{B_\alpha:t^*\lhd\pi(\alpha)\}$ and $\bigcup\{B_\alpha:t_{\beta_1}\lhd\pi(\alpha)\}$ are disjoint elements with small complements and that is the contradiction which proves the claim. Hence $\{t_\beta:\beta<\nu\}$ is a branch in T of length ν which contradicts the definition of T.

We have examined the first four cases and so we have constructed a branch in the tree of height ω. This branch has the property that the first coordinate of each of its nodes is a partition of X into small subsets of X. There are no infinite branches in the tree and so there is a highest node. By the construction of the tree, either this node invoked case 5 or there is a cardinal less than κ to which none of the five possibilities apply. If the former is true, then the first coordinate of this node is a countable partition of X into small subsets of X contradicting the countable completeness of J. If the latter is true, then this cardinal is either a Ξ-cardinal or a weakly compact cardinal and the lemma is proved. If $V = L$ and κ is any regular cardinal, then there is a κ-Aronszajn tree unless κ is weakly compact. This implies that, if κ is regular, then there is a κ-Aronszajn tree unless κ is weakly compact in an inner model.

Third, we show that, if κ is equal to a weakly compact cardinal, the sufficient condition of Lemma 1 can be improved.

LEMMA 3. *If I is κ-completable and κ is weakly compact, then I is κ-extendible.*

PROOF. Lemma 2.4 of [9] shows this lemma when the cardinality of X is κ, but, in that proof, this assumption is not necessary.

Fourth, we show that, unless there is a measurable cardinal in an inner model, the sufficient condition of Lemma 3 cannot be improved.

LEMMA 4. *If I is κ-extendible and there are no measurable cardinals in an inner model, then I is κ-completable.*

To facilitate the proof of Lemma 4, we state and prove a preliminary lemma:

LEMMA 5. *If there is a κ^+-extendible ideal on κ, then there is a measurable cardinal in an inner model.*

PROOF. Let K be the core model of Dodd and Jensen [6]. K is a model of the generalized continuum hypothesis, and so $|P^K(\kappa)| = (\kappa^+)^K \leqq \kappa^+$. If F is the dual of the κ^+-extendible ideal on κ of the hypothesis of the lemma, let $G \supset F$ be a countably complete filter of κ which measures each of the subsets of κ which are elements of K. Let M be the ultrapower $(K^\kappa)^K/G$ (thus elements of M are equivalence classes of functions $f: \kappa \to K$ such that $f \in K$). There is an elementary embedding $j: K \to M$ since the fundamental theorem of ultraproducts holds for M. Since G is countably complete, M is well-founded and so there is an an isomorphism $\pi: M \to N$ where N is a transitive class. Then $\pi \circ j$ is an elementary embedding of K into a transitive class. By results of Dodd and Jensen, any elementary embedding of K into a transitive class is an elementary embedding of K into K, the covering lemma must fail for K and there must be a measurable cardinal in an inner model.

PROOF OF LEMMA 4. Suppose I is κ-extendible but fails to be κ-completable. I must fail to be μ^+-completable for some cardinal $\mu < \kappa$. I is, however, μ^+-extendible. Thus we can assume without loss of generality that κ *is* a successor cardinal μ^+. I fails to be μ^+-completable and so there is a partition of the base set X into μ sets $\{A_\alpha\}_{\alpha<\mu}$ each of which is in I. Let $\pi : X \to \mu$ be defined by $\pi(x) = \alpha$ if and only if $x \in A_\alpha$. $\{X - \pi''(A) : X - A \in I\}$ is a μ^+-extendible ideal on μ and, by Lemma 5, we are done.

Fifth, we use Lemma 4 to improve Lemma 2 under the assumption that there are no measurable cardinals in an inner model.

LEMMA 6. *If I is κ-extendible and there are no measurable cardinals in an inner model then I is $(\kappa^\omega)^+$-completable unless κ is* equal *to either a weakly compact cardinal or a Ξ-cardinal.*

PROOF. Construct a tree of height three by taking the first three levels of the tree of height ω constructed in the proof of Lemma 2 and changing the second coordinate of the nodes of the third level. This is possible since the first inductive stage of the construction of the tree in the proof of Lemma 2 is possible unless κ is *equal* to a weakly compact cardinal or a Ξ-cardinal. The first coordinate of each node of the third level is a partition $\{A_\alpha\}_{\alpha<\mu}$ of the base set where $\mu < \kappa$. Let the second coordinate of this node be $\{\cup\{A_\alpha : \alpha \in X\} : X \subset \mu \text{ and } X \in K\}$ where K is the core model. This set has cardinality $(2^\mu)^K = (\mu^+)^K \leqq \mu^+ \leqq \kappa$ as required. As in the proof of Lemma 2, let J be a countably complete ideal which measures each element of the second coordinate of each node of the tree. As before, there must be a node of the third level whose first coordinate is a partition of the base set into small sets. Identifying these sets with points as in the proof of Lemma 4, we obtain a countably complete ideal which measures each subset of μ which is in K, and as in the proof of Lemma 5, there must be a measurable cardinal in an inner model.

THEOREM 1. *I is κ-extendible if and only if I is $(\kappa^\omega)^+$-completable unless κ is greater than or equal to either a weakly compact cardinal or a Ξ-cardinal.*

PROOF. Lemma 1 and Lemma 2.

COROLLARY. *I is ω-extendible if and only if I is $(2^{\aleph_0})^+$-completable.*

THEOREM 2. *If there are no measurable cardinals in an inner model and I is ω-extendible, and κ is not a Ξ-cardinal, then I is κ-extendible if and only if*

(a) *I is κ-completable (κ weakly compact),*

(b) *I is* κ^+*-completable* (κ *successor or* κ *singular of uncountable cofinality or* κ *regular limit but not weakly compact in L or* κ *can be obtained by nontrivial cardinal exponentiation or* κ *inaccessible but not weakly compact*),

(c) *I is* κ^{++}*-completable* (κ *singular of countable cofinality*).

PROOF. (a) is shown by Lemma 2.

By Lemma 6 and Lemma 1, under the hypothesis of this theorem, and when κ is not weakly compact, I is κ-extendible if and only if I is $(\kappa^\omega)^+$-completable. If there are no measurable cardinals in an inner model, the singular cardinals hypothesis holds and the cardinal arithmetic simplifies so that if κ has uncountable cofinality $(\kappa^\omega)^+ = (2^\omega)^+ \cdot \kappa^+$ and if κ has countable cofinality $(\kappa^\omega)^+ = (2^\omega)^+ \cdot \kappa^{++}$. By Theorem 1, if I is ω-extendible then I is $(2^\omega)^+$-completable and so the additional assumptions of (b) and (c) are sufficient (and necessary).

Ξ-cardinals have the consistency strength of weakly compact cardinals and assuming a slightly greater consistency strength Ξ-cardinals can behave like weakly compact cardinals with respect to completability of ideals:

THEOREM 3. *Let* κ *be a* Ξ*-cardinal and assume that there are no measurable cardinals in an inner model.*

(a) κ^+*-completable implies* κ*-extendible implies* κ*-completable.*

(b) *The existence of a* κ*-extendible ideal which is not* κ^+*-completable is consistent with the axioms of set theory relative to the consistency of the existence of an ineffable cardinal.*

PROOF. (a) By Lemma 1 and Lemma 4 (κ is greater than the continuum).

(b) By 2.1.3 of [4], if κ is an ineffable cardinal, then adding κ Cohen subsets of ω_1 yields a model of set theory where:

(1) κ is a Ξ-cardinal,

(2) if Σ is a subset of $L_{\kappa\kappa}$ of cardinality κ which has no model, then there is a Σ' of cardinality less than κ contained in Σ which has no model.

To see $[\kappa]^{<\kappa}$ is κ-extendible let $\{A_\alpha : \alpha < \kappa\}$ be subsets of κ. Let I be a unary predicate. Let Σ be the set of sentences which state that, for each $\alpha < \kappa$, $A_\alpha \in I$ or $\kappa - A_\alpha \in I$ and that I is a κ-complete ideal which contains $[\kappa]^{<\kappa}$. Each subset of Σ of cardinality less than κ has a model since, by Lemma 1, $[\kappa]^{<\kappa}$ is λ-extendible for each $\lambda < \kappa$. By (2), Σ has a model and the claim is proved.

A natural question is:

QUESTION 1. Does the consistency of the existence of an ineffable cardinal imply the consistency of the existence of a cardinal κ which is not weakly compact such that each κ-completable ideal is κ-extendible?

QUESTION 2. Does the consistency of the existence of a weakly compact cardinal imply the consistency of the existence of a cardinal κ which is not weakly compact and a κ-extendible ideal which is not κ^+-completable?

If there are measurables in an inner model then extendibility can exceed completability: if κ is measurable, then there is a κ^+-extendible ideal which is not κ^+-completable; if κ is compact, then each κ-completable ideal is κ^+-extendible.

Boban Veličković has shown that these situations can occur even when κ is not measurable or κ is not compact (of course κ is measurable in an inner model).

PROPOSITION 1. *If it is consistent that there is a supercompact cardinal, then it is consistent that there is a cardinal κ which is not measurable and a κ-completable ideal on κ which is κ^+-extendible.*

PROOF. (supplied by John Merrill). Let κ be κ^+-supercompact. Use a reverse Easton forcing extension (i.e. an iteration) to add α^+ Cohen subsets of α to each strongly inaccessible $\alpha \leqq \kappa$. Let this partial order be $\mathbf{P}$. Let $j : V \to \mathcal{M}$ be the ultrapower embedding and let $j(\mathbf{P}) = \mathbf{P} * Q$. The least inaccessible in $\mathcal{M}$ greater than κ is greater than κ^{++} (since $\mathcal{M}$ is closed under κ^+ sequences) and so Q is κ^{++}-closed. j can be extended to an elementary embedding $\tilde{j} : V^{\mathbf{P}} \to \mathcal{M}^{j(\mathbf{P})}$ (this is non-trivial: see p. 86 of [16]). Define an unltrafilter U on κ in $V^{j(\mathbf{P})}$ by $A \in U$ iff $\kappa \in \tilde{j}(A)$. U is well-defined since $A \subset \kappa$ and $A \in V^{j(\mathbf{P})}$ implies $A \in V^{\mathbf{P}}$ (Q is κ^+-closed). U is a κ-complete ultrafilter on κ by elementarity because $\tilde{j}$ is the identity on cardinals less than κ. This shows that κ is measurable in $V^{j(\mathbf{P})}$. By Σ^1_1-reflection (see lemma 4 of [11]) and the κ^+-closure of Q, κ is also measurable in $V^{\mathbf{P}}$. Let $\mathbf{R}$ be the partial order adding κ^{++}-Cohen subsets of κ. In $V^{\mathbf{P}*\mathbf{R}}$, κ is not measurable (since GCH holds at each inaccessible below κ but not at κ) but $[\kappa]^{<\kappa}$ is κ^+-extendible. To see this, let $\{A_\alpha : \alpha \in \kappa^+\} \in V^{\mathbf{P}*\mathbf{R}}$ be subsets of κ. There is a $S \subset \mathbf{R}$ which adds κ^+ Cohen subsets of κ such that $\{A_\alpha : \alpha \in \kappa^+\} \in V^{\mathbf{P}*S}$. The fact that $\mathbf{P} * S$ is isomorphic to $\mathbf{P}$ implies that κ is measurable in $V^{\mathbf{P}*S}$ and so $[\kappa]^{<\kappa}$ can be extended to a κ-complete ultrafilter which measures $\{A_\alpha : \alpha \in \kappa^+\}$.

QUESTION 3. Does the consistency of the existence of a measurable cardinal imply the consistency of the existence of a cardinal κ which is not measurable such that $[\kappa]^{<\kappa}$ is κ^+-extendible?

PROPOSITION 2 (Velickovic). *If it is consistent that there is a supercompact* *cardinal, then it is consistent that there is a cardinal κ which is not compact such that each κ-completable ideal is κ^+-extendible.*

PROOF (supplied by John Merrill). Let κ be a κ-directed closed indestructible supercompact (see [15]). Let $\mathbf{P} = \{E \in [\kappa^{++}]^{\leqq \kappa^+}$: E consists of ordinals of countable cofinality and is nonstationary in each initial segment$\}$ ordered by end-extension. Let S be the union of the generic filter of $\mathbf{P}$. In $V^{\mathbf{P}}$, let $Q = \{X \in [\kappa^{++}]^{\leqq \kappa^+} : X$ is closed and $X \cap S = \emptyset\}$ ordered by end-extension. Let $D = \{(E, \check{X}) : \sup E = \max X\}$. We claim that D is a κ-directed closed dense subset of $\mathbf{P} * Q$. To see that D is κ-directed closed, note that directed subsets of D are well-ordered and

$$(\bigcup \{E_\alpha : \alpha \in \kappa\}, (\check{\bigcup} \{X_\alpha : \alpha \in \kappa\}) \cup \{\sup \check{\bigcup} \{X_\alpha : \alpha \in \kappa\}\})$$

is a condition whenever $\{(E_\alpha, \check{X}_\alpha) = \alpha \in \kappa\}$ are conditions ($\bigcup \{E_\alpha : \alpha \in \kappa\}$ cannot be stationary in its supremum since $\bigcup \{X_\alpha : \alpha \in \kappa\}$ is closed unbounded in that ordinal). We claim that in $V^{\mathbf{P}}$, κ is not compact (because κ^{++} has an E-set: see [9] page 122) but that any κ-complete ideal on κ is κ^+-extendible. To see this, let $I \in V^{\mathbf{P}}$ be a κ-complete ideal on λ and let $\{A_\alpha : \alpha \in \kappa^+\}$ be subsets of λ. Work in $V^{\mathbf{P}*Q}$ to get a $f : \kappa^+ \to 2$ such that $I \cup \{A_\alpha^{f(\alpha)} : \alpha \in \kappa^+\}$ generates a κ-complete ideal. Q is κ^{++}-distributive so $f \in V^{\mathbf{P}}$. $I \cup \{A_\alpha^{f(\alpha)} : \alpha \in \kappa^+\}$ must also generate a κ-complete ideal in $V^{\mathbf{P}}$.

Part Two: When is an ideal (κ, λ)-extendible?

Section One: General results

This is a complicated question and so, to get a perspective, we shall emphasize the ideals $[\mu]^{<\nu}$.

First, we present some monotonicity results.

LEMMA 7. *If F is (κ, λ)-extendible, then F' is (κ', λ')-extendible whenever $\kappa' \geqq \kappa$, $\lambda' \leqq \lambda$, $F' \subseteq F$.*

LEMMA 8. *If $[\mu]^{<\nu}$ is (κ, λ)-extendible, then $[\mu']^{<\nu'}$ is (κ, λ)-extendible whenever $\mu' \geqq \mu$, $\nu' \leqq \nu$.*

Second, we shall demarcate the parameters.

LEMMA 9. *If F is λ-extendible, then F is (λ, λ)-extendible.*

LEMMA 10. *If $\nu = (2^\mu)^+$, then $[\mu]^{<\mu}$ is (ν, ν)-extendible.*

Third, we shall prove a lemma which shows that the question of whether $[\mu]^{<\nu}$

is (κ,ω)-extendible is equivalent to a problem in polarized partition relations [20].

LEMMA 11. $\binom{\kappa}{\mu}\rightarrow\binom{\lambda}{\nu}_2$ *implies* $[\mu]^{<\nu}$ *is* (κ,λ)*-extendible and the converse holds when* $\lambda=\omega$.

PROOF. Let $\{A_\xi:\xi\in\kappa\}$ be subsets of μ. Let $A\subset\kappa\times\mu$ be defined by $(\xi,\alpha)\in A$ if and only if $\alpha\in A_\xi$. The polarized partition relation can now be used to find $X\in[\kappa]^\lambda$ and $Y\in[\mu]^\nu$ such that either $X\times Y\subset A$ or $(X\times Y)\cap A=\varnothing$. In the first case, $\cap\{A_\xi:\xi\in X\}\supset Y$ and hence $\{\mu-A_\xi:\xi\in X\}$ can be extended to a σ-ideal which extends $[\mu]^{<\nu}$. If $(X\times Y)\cap A=\varnothing$, then $\{A_\xi:\xi\in X\}$ can be extended to such an indeal.

The second statement follows from the fact that if $A\subset\kappa\times\mu$ then extendibility can be applied to $\{A_\xi:\xi\in\kappa\}$ where $A_\xi=\{\alpha\in\mu:(\xi,\alpha)\in A\}$. Now (κ,ω)-extendibility yields $X\in[\kappa]^\omega$ such that either the set of sets indexed by X or the set of the complements can be extended to a σ-ideal. Either $X\times(\cap\{A_\xi:\xi\in X\})$ or $X\times(\cap\{\mu-A_\xi:\xi\in X\})$ will satisfy the polarized partition relation.

Fourth, we shall prove a lemma which shows that a saturation property of Laver [13] implies extendibility.

LEMMA 12. (κ,λ,ω)*-saturated ideals are* (κ,λ)*-extendible.*

PROOF. Suppose that I is an ideal on μ and that I is (κ,λ,ω)-saturated. Let $\{A_\xi:\xi\in\kappa\}$ be subsets of μ. If λ of these sets belong to I then we are done, so we may assume that none of these sets A_ξ belong to I. An application of the saturation property yields the result.

Section Two: $[\omega_1]^{<\omega_1}$

With these lemmata, we begin with an analysis of the specific question: When is $[\omega_1]^{<\omega_1}$ (κ,λ)-extendible?

There is the strongest possible negative consistency result:

THEOREM 4 (Hajnal, Juhasz). *It is consistent with any cardinal arithmetic that* $[\omega_1]^{<\omega_1}$ *is* not $(2^{\omega_1},\omega)$*-extendible.*

By Lemma 10, this is the best possible result. It was obtained by Hajnal and Juhasz [8] who showed more: that it is consistent with any cardinal arithmetic that $[2^\omega]^{<\omega_1}$ is not $(2^{\omega_1},\omega)$-extendible.

Prikry and Devlin later obtained the same result in L [17]:

THEOREM 5 (Devlin, Prikry). *$V = L$ implies $[\omega_1]^{<\omega_1}$ is not $(2^{\omega_1}, \omega)$-extendible.*

In fact, Silver's $W(\kappa)$ implies that $[\omega_1]^{<\omega_1}$ is not (κ, ω)-extendible and so it is not possible to prove that it is consistent with $2^{\aleph_0} = \aleph_1$ that $[\omega_1]^{<\omega_1}$ is not (ω_2, ω)-extendible.

Cardinal arithmetic alone does imply a negative result of Sierpinski:

THEOREM 6 (Sierpinski). *CH implies $[\omega_1]^{<\omega_1}$ is not (ω_1, ω)-extendible.*

We conjecture that there are no limitations on a positive consistency result when $\lambda = \omega$ beyond that of Theorem 6.

One way of obtaining these results is the following:

LEMMA 13. *Let $B \in [{}^{\omega_1}2]^{\kappa}$. If there exist $g_\alpha : 2^\alpha \to 2$ $(\alpha \in \omega_1)$ such that, for any $F \in [B]^\omega$, there is $\alpha \in \omega_1$ such that, for any $\beta \geqq \alpha$, g_β is not constant on $\{f \restriction \beta : f \in F\}$, then $[\omega_1]^{<\omega_1}$ is not (κ, ω)-extendible.*

PROOF. Let $A_f = \{\alpha \in \omega_1 : g_\alpha(f \restriction \alpha) = 1\}$ $(f \in B)$. Claim that $[\omega_1]^{<\omega_1}$ cannot be extended by any infinite set of A_f's. If not, let $G \supset [\omega_1]^{<\omega_1}$ measure A_f $(f \in F)$ where $F \in [B]^\omega$. We can assume that either $A_f \in G$ $(f \in F)$ or $\omega_1 - A_f \in G$ $(f \in F)$. There is $\alpha \in \omega_1$ such that, for any $\beta \geqq \alpha$, g_β is not constant on $\{f \restriction \beta : f \in F\}$. If $\beta \in \bigcap \{A_f : f \in F\}$, then $g_\beta(f \restriction \beta) = 1$ $(f \in F)$. If $\beta \in \bigcap \{\omega_1 - A_f : f \in F\}$, then $g_\beta(f \restriction \beta) = 0$ $(f \in F)$. In either case, $\beta < \alpha$ and a countable intersection of elements of G is contained in α which is a contradiction.

PROOF OF THEOREM 4. Add $2^{\aleph_0}$ Cohen reals to any model V of set theory to get generic objects $g_\alpha : 2^\alpha \to 2$ $(\alpha \in \omega_1)$, let $B = ({}^{\omega_1}2 \cap V)$ and apply Lemma 13. If $F \in [B]^\omega$, then F is in an extension of V obtained by adding countably many generic objects. There is $\alpha \in \omega_1$ such that, for any $\beta \geqq \alpha$, $\{f \restriction \beta : f \in F\}$ is an infinite set in that extension. No g_β is constant on an infinite set over which it is generic.

PROOF OF THEOREMS 5 AND 6. Silver's axiom $W(\kappa)$ says that there is $B \in [{}^{\omega_1}2]^\kappa$ and a function S with domain ω_1 such that

(1) for each $\alpha \in \omega_1$, $S(\alpha)$ is a subset of $[{}^\alpha 2]^\omega$ of cardinality ω,

(2) for each $F \in [B]^\omega$, there is $\alpha \in \omega_1$ such that, for each $\beta \geqq \alpha$, $\{f \restriction \beta : f \in F\} \in S(\beta)$.

Construct each g_α by induction to be constant on no element of $S(\alpha)$ and apply Lemma 13. CH implies $W(\omega_1)$ and $V = L$ implies $W(\omega_2)$ while the consistency of the failure of $W(\omega_2)$ with CH can not be proved without assuming the existence of an inaccessible cardinal.

It is reasonable to ask whether Theorem 6 can be proved without assuming CH or, perhaps, by assuming only $2^{\aleph_0} < 2^{\aleph_1}$. The following theorem provides the answer to these questions.

THEOREM 7 (Miller and Velickovic). *It is consistent with any cardinal arithmetic in which CH fails that* $[\omega_1]^{<\omega_1}$ *is* (ω,ω)*-extendible.*

PROOF (see p.289 A10 of [12]). Start with any model V and iterate C.C.C. forcing with finite supports ω_2 times to form M_α $(\alpha \leqq \omega_2)$. U_α will be an ultrafilter on $P(\omega)$ in M_α. $M_{\alpha+1}$ is obtained by adjoining to M_α an $a_\alpha \subset \omega$ with $(\forall x \in U_\alpha)|a_\alpha - x| < \omega$. Claim that $[\omega_1]^{<\omega_1}$ is (ω,ω)-extendible in $V[G]$. Let $\{A_n : n \in \omega\}$ be a family of subsets of ω_1. $\{A_n : n \in \omega\}$ can be coded by a subset of ω_1 and so there is $\alpha \in \omega_2$ such that $\{A_n : n \in \omega\} \in M_\alpha$. Let $\{B_\gamma : \gamma \in \omega_1\}$ be a family of subsets of ω defined by $n \in B_\gamma$ iff $\gamma \in A_n$. U_α measures each B_γ. We assume that, for each $\gamma \in \omega_1$, $B_\gamma \in U_\alpha$. Therefore, for each $\gamma \in \omega_1$, $a_\alpha - B_\gamma \in [\omega]^{<\omega}$. There is $F \in [\omega]^{<\omega}$ and $X \in [\omega_1]^{\omega_1}$ such that $\gamma \in X$ implies $a_\alpha - B_\gamma \in F$. For each $\gamma \in X$, $B_\gamma \supset a_\alpha - F$ and so, for each $n \in \omega - F$, $A_n \supset X$.

This yields an application to set-theoretic topology which requires the following definition: $X \subseteq {}^{\omega_1}2$ is an *HFD* if and only if for each $\bar{X} \in [X]^{\aleph_0}$ there is $\xi \in \omega_1$ such that for each $f \in \bigcup\{{}^{\Gamma}2 : \Gamma \in [\omega_1 - \xi]^{<\aleph_0}\}$ there is $g \in \bar{X}$ such that $f \subseteq g$.

The next proposition shows that it is possible to obtain a model where $2^{\aleph_0} < 2^{\aleph_1}$ and there is no HFD. (It is still not known whether $2^{\aleph_0} < 2^{\aleph_1}$ implies that there is an S-space.)

THEOREM 8. *If* $[\omega_1]^{<\aleph_1}$ *is* $(\kappa,\aleph_0)$*-extendible then there is no HFD of size* κ.

PROOF. Let $\{f_\eta : \eta \in \kappa\} \subseteq {}^{\omega_1}2$. Choose J a σ-ideal extending $[\omega_1]^{<\aleph_1}$ such that there is some $\Gamma \in [\kappa]^{\aleph_0}$ such that for $\xi \in \Gamma$, $\{f_\xi^{-1}\{0\}, f_\xi^{-1}\{1\}\} \cap J \neq 0$. Without loss of generality assume that for $\xi \in \Gamma$, $f_\xi^{-1}\{0\} \in J$. Since J is countably complete, for each $\beta \in \omega_1$ there is $\alpha \in \omega_1 - \beta$ such that

$$\alpha \in \bigcap\{f_\xi^{-1}\{1\}; \xi \in \Gamma\}.$$

Hence $\{f_\xi : \xi \in \Gamma\}$ is not an HFD and so neither is $\{f_\eta ; \eta \in \kappa\}$.

Since $W(\kappa)$ implies the failure of the (ω_2,ω)-extendability of $[\omega_1]^{<\omega_1}$ and the consistency of the failure of $W(\omega_2)$ requires an inaccessible cardinal, it is reasonable to conjecture that the existence of an inaccessible cardinal implies the consistency of the (ω_2,ω)-extendibility of $[\omega_1]^{<\omega_1}$. The role of ω_2 is crucial since it will be shown in Theorem 12 that the (ω_3,ω)-extendibility of $[\omega_1]^{<\omega_1}$ is consistent with any cardinal arithmetic and that large cardinals are not needed for this result. One might also ask whether the fact that $[\omega_1]^{<\omega_1}$ is (ω_2, ω)-extendible implies the consistency of a large cardinal.

The polarized partition relation translation of the question of the (ω_2, ω)-extendibility of $[\omega_1]^{<\omega_1}$ was asked by Erdos, Hajnal and Rado in 1965 [7] and with GCH by Laver in 1980 [14].

These positive consistency results do not differentiate between (ω, ω)-extendibility and (ω_1, ω)-extendibility, on the one hand, and (ω_2, ω)-extendibility and $(2^{\omega_1}, \omega)$-extendibility, on the other hand. The first is, surprisingly, not a coincidence:

THEOREM 9. *$[\omega_1]^{<\omega_1}$ is (ω, ω)-extendible if and only if $[\omega_1]^{<\omega_1}$ is (ω_1, ω)-extendible.*

PROOF. We prove one direction. Let $\{C_n : n \in \omega\}$ be a counterexample to (ω, ω)-extendibility and define $B_\xi = \{n \in \omega : \xi \in A_n\}$. Also, let $\{A_\alpha : \alpha \in \omega_1\}$ be a strictly increasing (mod finite) sequence of subsets of ω and choose $\{f_\alpha : \alpha \in \omega_1\}$ to satisfy the conclusion of the following lemma:

LEMMA 14. *If $\{A_\alpha : \alpha \in \omega_1\}$ is a strictly increasing (modulo finite sets) sequence of subsets of ω then there is a sequence $\{f_\alpha : \alpha \in \omega_1\}$ such that:*

(0) *if $\alpha \in \omega_1$ then $f_\alpha : \omega \to \omega_1$ is* 1-1,

(1) *if $\alpha \in \beta \in \omega_1$ then $\alpha \subseteq f''_\alpha A_\alpha \subseteq f''_\beta A_\beta$,*

(2) *if $\alpha \in \beta \in \omega_1$ then there is $k \in \omega$ such that $f_\alpha \restriction (A_\alpha - k) = f_\beta \restriction (A_\alpha - k)$.*

PROOF. If $\{f_\beta : \beta < \alpha\}$ are defined, define $\{f_\alpha(n) : n \in \omega\}$ by induction upward on $k \in A_\alpha$. List $\alpha = \{\alpha_i ; i \in \omega\}$.

Let P_n be the condition: If there is $i \leqq n$ such that $k \in A_{\alpha_i}$, then let i be minimal and let $f_\alpha(k) = f_{\alpha_i}(k)$.

Let Q be the condition: let i be minimal such that $f_\alpha(l) \neq \alpha_i$ $(l < k)$ and let $f_\alpha(k) = \alpha_i$. The induction priority is to apply P_j to all integers between k_{j-1} and the least $k_j > k_{j-1}$ to which it does not apply and to apply Q to k_j.

Condition Q is applied infinitely-many times because $\{A_\alpha : \alpha \in \omega_1\}$ is strictly almost increasing and so each P_n is applied. For each n, there is $m \in \omega$ such that $\{f_{\alpha_i} \restriction A_{\alpha_i} - m : i \leqq n\}$ are compatible. Therefore, for $k \geqq \max\{m, k_n\}$, when $f_\alpha(k)$ is decided, P_j applies where $j \geqq n$. If $k \in A_{\alpha_n}$, then letting i be minimal such that $k \in A_{\alpha_i}$, $k \geqq m$ implies

$$f_{\alpha_n}(k) = f_\alpha(k).$$

Extend f_α to domain ω arbitrarily.

CONTINUING THE PROOF OF THEOREM 9. For $\beta \in \omega_1$, let $X_\beta = \{\alpha \in \omega_1 : f_\alpha^{-1}(\beta) \in B_\alpha\}$. We show that $\{X_\beta : \beta \in \omega_1\}$ is a counterexample to (ω_1, ω)-extendibility. Suppose not, that $\Gamma \in [\omega_1]^\omega$ and $|\cap\{X_\beta : \beta \in \Gamma\}| = \omega_1$ (for example). Let $\cap\{X_\beta : \beta \in \Gamma\} = S$. Then $\alpha \in S$ and $\beta \in \Gamma$ implies $\alpha \in X_\beta$ and $f_\alpha^{-1}(\beta) \in B_\alpha$ and $\alpha \in C_{f_\alpha^{-1}(\beta)}$. Let $\alpha_0 = \inf(S - \Gamma)$. By (2), $f''_{\alpha_0}(A_{\alpha_0}) \supset \Gamma$. Let $f_{\alpha_0} \cap (\omega \times \Gamma) = f$. For each $\alpha \in S - \Gamma$, f_α almost contains f. There is an uncountable $T \subset S - \Gamma$ and a finite $\sigma \subset f$ such that $\alpha \in T$ implies $f_\alpha \supset g - \sigma$. Then $\alpha \in T$ and $\beta \in \operatorname{rng}(g)$ implies $\alpha \in C(g-\sigma)^{-1}(\beta)$ and so $\cap\{C_n : n \in \operatorname{dom}(g-\sigma)\} \supset T$.

This proof shows that the existence of a countable HFD implies the existence of an uncountable HFD, answering a question of Juhasz.

THEOREM 10. *If there is a countable HFD, then there is an uncountable HFD.*

PROOF. The proof is similar to the proof of Theorem 9.

Note that the translation of this result into polarized partition relations is a new result.

What about positive consistency results when λ is uncountable?

One positive consistency result is a consequence of Lemma 12 and a result of Laver [13]:

THEOREM 11 (Laver). *If it is consistent that there is a huge cardinal, then it is consistent that GCH holds and $[\omega_1]^{<\omega_1}$ is (ω_2, ω_2)-extendible.*

More generally, if it is consistent that there is a huge cardinal and if κ is a regular cardinal, then it is consistent that GCH holds and $[\kappa^+]^{<\kappa^+}$ is $(\kappa^{++}, \kappa^{++})$-extendible.

A positive consistency result without the assumption of large cardinals is:

THEOREM 12. *If $\aleph_3$ Cohen subsets of ω_1 are added to a model of GCH, then $[\omega_1]^{<\omega_1}$ is (ω_3, ω_3)-extendible.*

PROOF. Let $P = Fn(\omega_3, 2, \omega_1)$. CH implies that no cardinals are destroyed in the extension $V[G]$. Let τ be a name, as defined in [12], and p a condition in G such that $p \Vdash$ "τ is a sequence of subsets of ω_1 indexed by ω_3". Let $\{n_\alpha : \alpha \in \omega_3\}$ be such that

(1) For each $\alpha \in \omega_3$, $n_\alpha = \cup\{\{\gamma\} \times A_\alpha(\gamma) : \gamma \in \omega_1\}$ where, for each $\gamma \in \omega_1$, $A_\alpha(\gamma)$ is an antichain in P, and

(2) $p \Vdash$ "$\tau = \{n_\alpha : \alpha \in \omega_3\}$". For each α, let $\Sigma_\alpha = \cup\{\operatorname{dom}'' A_\alpha(\gamma) : \gamma \in \omega_1\}$. Now $|\Sigma_\alpha| \leqq \aleph_1$ and so $2^{\omega_1} = \omega_2$ implies that we can assume, without loss of generality, that $\{\Sigma_\alpha : \alpha \in \omega_3\}$ is a Δ-system with root Δ. We can assume, without loss of

generality, that $\Delta = 0$ (letting $P_\Delta = Fn(\Delta, 2, \omega_1)$ and $P_{-\Delta} = Fn(\omega_3 - \Delta, 2, \omega_1)$, $|\Delta| \leqq \aleph_1$ implies that $V[G \cap P_\Delta] \models$GCH; we can modify each $A_\alpha(\gamma)$ to be an antichain in $P_{-\Delta}$ by subtracting conditions which are incompatible with $G \cap P_\Delta$ and restricting each remaining condition to $\omega_3 - \Delta$; and so work in $V[G \cap P_\Delta]$ with $P_{-\Delta}$). We assume, without loss of generality, $\{n_\alpha : \alpha \in \omega_3\}$ are isomorphic. That is, letting $\psi_\alpha : \Sigma_\alpha \to \omega_1$ be an injection, for each $\alpha \in \omega_3$, we assume that $p \in A_\alpha(\gamma)$ if and only if $p \circ \psi_\alpha^{-1} \circ \psi_\beta \in A_\beta(\gamma)$ whenever $\alpha \in \omega_3$, $\beta \in \omega_3$, and $\gamma \in \omega_1$ (there are $\aleph_2$ possibilities). If there exists $q \leqq p$, $\alpha \in \omega_3$ and $B \in [\omega_1]^{\omega_1}$ such that $q \Vdash$"$B \subseteq n_\alpha$" then assume, without loss of generality, that $\mathrm{dom}(q) \subset \mathrm{dom}(p) \cup \Sigma_\alpha$. Any condition below p may be extended to force $B \subset n_\beta$ and this implies that p forces that, for $\aleph_3$-many β, $B \subset n_\beta$ as required. Otherwise p forces that no n_α contains an uncountable set in V. That is, for each $q \leqq p$ and $\alpha \in \omega_3$, there is $\beta(\alpha, q)$ such that, for each $\gamma > \beta(\alpha, q)$, there is $r \leqq q$ such that $r \Vdash$"$\gamma \notin n_\alpha$" (we can assume, without loss of generality, that $\mathrm{dom}(r) \subset \mathrm{dom}(q) \cup \Sigma_\alpha$). Fix $\Omega \in [\omega_3]^\omega$. Whenever $\beta \in \omega_1$ and $q \leqq p$, we can find $\gamma > \beta$ such that $\gamma > \beta(\alpha, q)$ for each $\alpha \in \Omega$, and so there is $r \leqq q$ such that $r \Vdash$"$\gamma \in \bigcap \{\omega_1 - n_\alpha : \alpha \in \Omega\}$". This implies that $p \Vdash$"$|\bigcap\{\omega_1 - n_\alpha : \alpha \in \Omega\}| = \aleph_1$" as required.

Since Theorem 9 shows that as far as $[\omega_1]^{<\omega_1}$ is concerned, there is no difference between (ω_1, ω)-extendability and (ω, ω)-extendability, it might be tempting to conjecture that there is also no difference between (κ, ω)-extendability and (κ^+, ω)-extendability, or perhaps even between $(2^{\aleph_1}, \omega)$-extendability and (ω, ω)-extendability. The next result shows that this is not so.

THEOREM 13. *It is consistent with any cardinal arithmetic in which* $2^\omega = \omega_1$ *that* $[\omega_1]^{<\omega_1}$ *is* (ω_3, ω)*-extendible but not* (ω_2, ω)*-extendible.*

PROOF. Add Cohen subsets of ω_1 to a model of $V = L$, and apply Theorems 5 and 12.

B. Velickovic has noted that by starting with a model where $W(\kappa)$ hold and adding κ^+ Cohen subsets of ω_1, it is possible to get (κ^+, ω)-extendability without (κ, ω)-extendibility.

QUESTION 4. Does the (ω_2, ω_2)-extendibility of $[\omega_1]^{<\omega_1}$ imply the consistency of large cardinals?

Establishing the consistency of (ω_1, ω_1)-extendibility seems to be difficult. Todorčević has noticed the following:

THEOREM 14 (Todorčević). $[\omega_1]^{<\omega_1}$ *is* (ω_1, ω_1)*-extendible iff* $\omega_1 \to (\omega_1; \omega_1)^2_2$.

We need a definition to understand this: $\omega_1 \to (\omega_1; \omega_1)^2_2$ if and only if, for every $F:[\omega_1]^2 \to 2$, there are uncountable subsets of ω_1, A and B and $i \in 2$ such that, if $\alpha \in A$ and $\beta \in B$ and $\alpha > \beta$ then $F(\{\alpha, \beta\}) = i$.

PROOF OF THEOREM 14. First suppose that $\omega_1 \to (\omega_1; \omega_1)^2_2$. Given $\{A_\alpha : \alpha \in \omega_1\}$ subsets of ω_1, define $F(\{\alpha, \beta\})$ to be 1 if $\alpha \in A_\beta$ and 0 otherwise. Then find A, B and i satisfying the definition of the partition relation. If $i = 1$, then for any $\gamma \in \omega_1$, $A - \gamma \subset \bigcap \{A_\beta : \beta \in B \cap \gamma\}$ and hence $\{A_\beta : \beta \in B\}$ generates a countably complete filter. If $i = 0$, then $\{\omega_1 - A_\beta : \beta \in B\}$ generates a countably complete filter.

Now suppose that $[\omega_1]^{<\omega_1}$ is (ω_1, ω_1)-extendible and let $F:[\omega_1]^2 \to 2$. Let $A_\alpha = \{\beta \in \omega_1 : F(\{\alpha, \beta\}) = 1\}$. Let $\bar{B} \in [\omega_1]^{\omega_1}$ be such that either $\{A_\alpha : \alpha \in \bar{B}\}$ or $\{\omega_1 - A_\alpha : \alpha \in \bar{B}\}$ generates a countably complete filter. Define $B \in [\bar{B}]^{\omega_1}$, $A \in [\omega_1]^{\omega_1}$ inductively so that $\alpha \in A$ implies $\alpha \in \bigcap \{A_\beta : \beta \in B \cap \alpha\}$ (or $\alpha \in \bigcap \{\omega_1 - A_\beta : \beta \in B \cap \alpha\}$). If $\alpha \in A$, $\beta \in B$ and $\alpha > \beta$ then $\beta \in A_\alpha$ (or $\beta \notin A_\alpha$) and $F(\{\alpha, \beta\}) = 1$ (or $F(\{\alpha, \beta\}) = 0$).

In December 1984 Todorcevic announced:

THEOREM 15 (Todorcevic) (ZFC). *$[\omega_1]^{<\omega_1}$ is not (ω_1, ω_1)-extendible.*

There are many other open questions on the (κ, λ)-extendibility of $[\omega_1]^{<\omega_1}$ when λ is uncountable but we content ourselves with:

QUESTION 5. Does the fact that $[\omega_1]^{<\omega_1}$ is (ω_2, ω_1)-extendible imply that $[\omega_1]^{<\omega_1}$ is (ω_2, ω_2)-extendible?

Section Three: Arbitrary σ-ideals

Moving from the specific to the general, we continue with an analysis of the (κ, λ)-extendibility of arbitrary σ-ideals. There are no positive consistency results here at all!

THEOREM 16. *ZFC implies that there is a σ-ideal on 2^ω which is not $(2^{(2^\omega)}, \omega)$-extendible. In fact a stronger result is true: ZFC implies that, for each cardinal κ which is such that $\kappa^\omega = \kappa$, there is an ideal on κ which is not $(2^\kappa, \omega)$-extendible.*

PROOF. $\kappa^\omega = \kappa$ implies that there is $D \subset 2^{(2^\kappa)}$ which has size κ and is a dense subset when ${}^{(2^\kappa)}2$ has the G_δ-topology [5].

Since every set of measure 0 is contained in the countable union of sets of the form $\{f \in {}^{(2^\kappa)}2 : f \supset g\} = [g]$, where g is countable and infinite, a diagonalization argument shows that if X has measure zero then there is a countable and infinite

g such that $X \cap [g] = \emptyset$. Since $[g]$ is a G_δ-set it follows that D cannot have measure zero. Therefore, if $I = \{X \cap D : X \subset 2^{(2^\kappa)}$ and X has measure zero$\}$, then I is a countably complete ideal on D. Let $A^i_\alpha = \{f \in D : f(\alpha) = i\}$ for $\alpha \in {}^\kappa 2$ and $i \in 2$. By the density of D, $\alpha \neq \beta$ implies $A^0_\alpha \neq A^0_\beta$. For any subset $\Gamma \in [2^\kappa]^\omega$ and $h : \Gamma \to 2$, $\cap\{A^{h(\gamma)}_\gamma : \gamma \in \Gamma\} \in I$ and so $\{A^0_\alpha : \alpha \in 2^\kappa\}$ witnesses that I is not $(2^\kappa, \omega)$-extendible.

There are at least two possible directions in which to continue:

First, we can examine the (κ, λ)-extendibility of specific ideals such as $[\mu]^{<\nu}$ and, second, we can examine the (κ, λ)-extendibility of restricted classes of ideals.

The second of these directions can be realized in the investigation of two questions. The first is motivated by trying to avoid Theorem 16 by restraining the cardinality of the underlying set:

(1) When is an arbitrary σ-ideal on a set of cardinality less than 2^ω, (κ, λ)-extendible?

The second is motivated by trying to avoid Theorem 16 by prescribing a greater completeness.

(2) When is an arbitrary μ-complete ideal (κ, λ)-extendible?

Section Four: $[2^{\omega_1}]^{<2^{\omega_1}}$

With the first direction in mind, we conclude with an analysis of the specific question: When is $[2^{\omega_1}]^{<2^{\omega_1}}$ (κ, λ)-extendible?

To simplify the discussion, we avoid the pathology of cardinals of countable cofinality by assuming the singular cardinals hypothesis.

First, cardinal arithmetic implies a positive result:

LEMMA 15. *If* $2^\omega < 2^{\omega_1}$, *then* $[2^{\omega_1}]^{<2^{\omega_1}}$ *is* (μ, μ)-*extendible whenever* $\mu < 2^{\omega_1}$.

Second, when $2^\omega = 2^{\omega_1}$, we have the strong negative consistency result mentioned in the remarks following Theorem 4: It is consistent with any cardinal arithmetic in which $2^\omega = 2^{\omega_1}$ that $[2^{\omega_1}]^{<2^{\omega_1}}$ is not $(2^{\omega_1}, \omega)$-extendible.

This is not the strongest possible negative consistency result and so we have:

QUESTION 6. Does ZFC imply that $[2^{\omega_1}]^{<2^{\omega_1}}$ is $(2^{(2^{\omega_1})}, \omega)$-extendible?

Third, when $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, we have the weak negative consistency result:

THEOREM 17. *It is consistent with* $2^\omega = \omega_1$ *and* $2^{\omega_1} = \omega_2$ *that* $[2^{\omega_1}]^{<2^{\omega_1}}$ *is not* $(2^{\omega_1}, 2^{\omega_1})$-*extendible.*

This follows from the following more general result which is new when $2^{\omega} < 2^{\omega_1}$:

Suppose κ and λ are regular cardinals and $\kappa < \lambda$. Then it is consistent with ZFC that $[\kappa^+]^{<\kappa}$ is not $[\kappa^+, \kappa^+]$-extendible and $2^{\aleph_0} = 2^{2^{\aleph_0}} = \kappa$ and $2^{\kappa} = \lambda$.

Proof. Let $V \models$"MA and $2^{\aleph_0} = \kappa$ and $2^{\kappa} = \lambda$". Define a partial order P by $p \in P$ if and only if $p = (A^p, \{f^p_\alpha; \alpha \in A^p\}, \mathscr{C}^p)$ where

(1) $A^p \in [\kappa^+]^{<\kappa}$;

(2) if $\alpha \in A^p$ then $f^p_\alpha : A^p \to 2$;

(3) $|\mathscr{C}^p| < \kappa$;

(4) if $(g, X) \in \mathscr{C}^p$

(a) the order type of $D(g)$ is a countable limit ordinal,

(b) $D(g) \subseteq A^p$,

(c) $g''D(g) \subseteq 2$,

(d) $\cup \{(f^p_\beta)^{-1}\{g(\beta)\};\ \beta \in D(g)\} = A^p - X$,

(e) $X \subseteq (\cup D(g)) \cap A^p$.

The ordering on P is defined by $p \leqq q$ if and only if:

(5) $A^p \supseteq A^q$;

(6) $\mathscr{C}^p \supseteq \mathscr{C}^q$;

(7) if $\alpha \in A^q$ then $f^p_\alpha \restriction A^q = f^q_\alpha$.

Clearly, $(P, \leqq)$ is κ-closed. Since, in the ground model, $2^{\kappa} = \kappa$, the Δ-system lemma can be applied. Hence, to show that $(P, \leqq)$ satisfies the κ^+-chain condition, it suffices to show the following:

(8) if $\Phi : A^p \to A^q$ is an isomorphism of p and q such that $\Phi \restriction A^p \cap A^q$ is the identity map and $\cup (A^p \cap A^q) \in \cap (A^p - A^q)$ and $\cup A^p \in \cap A^q - A^p$ then p and q are compatible.

In order to prove (8) suppose that p, q and Φ are given. Let $A^r = A^p \cup A^q$ and $\mathscr{C}^r = \mathscr{C}^p \cup \mathscr{C}^q$. Functions $\{f^r_\alpha; \alpha \in A^r\}$ must be defined so that $f^r_\alpha : A^r \to 2$ and $f^r_\alpha \supseteq f^p_\alpha \cup f^q_\alpha$ for each $\alpha \in A^r$. Furthermore, the function f^r_α must be defined so that if $(g, X) \in \mathscr{C}^p \Delta \mathscr{C}^q$ then $\cup \{(f^r_\alpha)^{-1}\{g(\alpha)\}; \alpha \in D(g)\} = A^r - X$. Note that this will automatically hold if (g, X) is in $\mathscr{C}^p \cap \mathscr{C}^q$.

To construct the functions f^r_α let

$$S(g,X) = \begin{cases} D(g) - A^q & \text{if } (g,X) \in \mathscr{C}^p - \mathscr{C}^q, \\ D(g) - A^p & \text{if } (g,X) \in \mathscr{C}^q - \mathscr{C}^p. \end{cases}$$

Note that if $(g, X) \in \mathscr{C}^p \Delta \mathscr{C}^q$ then $D(g) \not\subseteq A^p \cap A^q$ for otherwise, by (4e), $D(g) \cup X \subseteq A^p \cap A^q$ and hence, by the properties of Φ, $(g, X) \in \mathscr{C}^p \cap \mathscr{C}^q$. Therefore, by (4a), $|S(g, X)| = \aleph_0$.

Now let $\Lambda = (A^q \times (A^p - A^q)) \cup (A^p \times (A^q - A^p))$ and let $Q = \cup\{{}^{\Gamma}2; \Gamma \in [\Lambda]^{<\aleph_0}\}$. Under reverse inclusion Q has the countable chain condition. therefore, since MA holds, it is possible to find $h \in {}^{\Lambda}2$ such that:

(9) if $\alpha \in A^p$ and $(g, X) \in \mathscr{C}^q$ then there is $\beta \in S(g, X)$ such that $h(\alpha, \beta) = g(\beta)$;

(10) if $\alpha \in A^q$ and $(g, X) \in \mathscr{C}^p$ then there is $\beta \in S(g, X)$ such that $h(\alpha, \beta) = g(\beta)$.

Now define

$$(11)\ f^r_\beta = \begin{cases} f^p_\beta \cup \{(\alpha, h(\alpha, \beta)); \alpha \in A^q - A^p\} & \text{if } \beta \in A^p - A^q, \\ f^q_\beta \cup \{(\alpha, h(\alpha, \beta)); \alpha \in A^p - A^q\} & \text{if } \beta \in A^q - A^p, \\ f^p_\beta \cup f^q_\beta & \text{if } \beta \in A^p \cap A^q. \end{cases}$$

Let $r = (A^r, \{f^r_\alpha; \alpha \in A^r\}, \mathscr{C}^r)$. It suffices to check that (4d) is satisfied by r. Let $(g, X) \in \mathscr{C}^q - \mathscr{C}^p$ and $\alpha \in A^r - X$. If $\alpha \in A^q$ then there is $\delta \in D(g)$ such that $\alpha \in (f^q_\delta)^{-1}\{g(\delta)\}$ and hence $\alpha \in (f^r_\delta)^{-1}\{g(\delta)\}$. If $\alpha \in A^p - A^q$ choose, using (9), $\beta \in S(g, X)$ such that $h(\alpha, \beta) = g(\beta)$. Then $f^r_\beta(\alpha) = h(\alpha, \beta)$ by (11) because $\alpha \in A^p - A^q$. Hence $\alpha \in (f^r_\beta)^{-1}\{g(\beta)\}$. A similar argument for the case when $(g, X) \in \mathscr{C}^p - \mathscr{C}^q$ finishes the proof that $(P, \leqq)$ satisfies the κ^+-chain condition.

Now let G be $(P, \leqq)$-generic over V. Let

$$f_\alpha = \cup\{f^p_\alpha; p \in G \text{ and } \alpha \in A^p\}.$$

It will be shown that $\{f_\alpha^{-1}\{1\}; \alpha \in \kappa^+\}$ witnesses that $[\kappa^+]^{<\kappa}$ is not (κ^+, κ^+)-extendible. Suppose that $p \in P$ and $p \Vdash$"$X \in [\kappa^+]^{\kappa^+}$ and $g \in {}^X2$". It must be shown that there is $q \leqq p$ and $\Gamma \in [\kappa^+]^{\aleph_0}$ and $Z \in [\kappa^+]^{<\kappa}$ such that

$$q \Vdash \text{"}Z \cup (\cup\{f_\alpha^{-1}\{g(\alpha)\}; \alpha \in \Gamma\}) = \kappa^+ \text{ and } \Gamma \subset X\text{"}.$$

To this end, use the κ-closure of $(P, \leqq)$ and the fact that $p \Vdash$"$X \in [\kappa^+]^{\kappa^+}$" to find $r \leqq p$ such that:

(12) if $\alpha \in A^r$ then $r \Vdash$"$\alpha \in X$" or $r \Vdash$"$\alpha \notin X$";

(13) $\{\alpha \in A^r; r \Vdash$"$\alpha \in X$"$\}$ is cofinal in A^r;

(14) if $r \Vdash$"$\alpha \in X$" then $r \Vdash$"$g(\alpha) = 0$" or $r \Vdash$"$g(\alpha) = 1$".

Using the ω_1-closure of $(P, \leqq)$ it is possible to insist that the order type of A^r is a limit ordinal of countable cofinality.

Define a function g^* so that $D(g^*)$ is a countable and cofinal subset of $\{\alpha \in A^r; r \Vdash$"$\alpha \in X$"$\}$ and such that if $\alpha \in D(g^*)$ then $r \Vdash$"$g(\alpha) = g^*(\alpha)$". Let

$$q = (A^r, \{f^r_\alpha; \alpha \in A^r\}, \mathscr{C}^r \cup \{(g^*, A^r)\}).$$

It is easy to check that $q \in P$ and $q \leqq r \leqq p$. But, since $(g^*, A') \in \mathscr{C}'$, it follows that

$$q \Vdash \text{“}A' \cup (\cup \{f_\alpha^{-1}\{g(\alpha)\}; \alpha \in D(g^*)\}) = \kappa^+\text{”}.$$

Furthermore, $q \Vdash$"$D(g^*) \subseteq X$" and hence q is the desired condition.

This leaves:

QUESTION 7. Is it consistent with any cardinal arithmetic that $[2^{\omega_1}]^{<2^{\omega_1}}$ is not $(2^{\omega_1}, 2^{\omega_1})$-extendible?

And, for example, in general,

QUESTION 8. Does ZFC imply that $[\kappa]^{<\kappa}$ is $(2^\kappa, 2^\kappa)$-extendible whenever $2^\kappa > 2^{\omega_1}$?

Fourth, we have some positive consistency results:

THEOREM 18 (Laver). *If it is consistent that there is a huge cardinal, then it is consistent that GCH holds and* $[2^{\omega_1}]^{<2^{\omega_1}}$ *is* $((2^{\omega_1})^+, (2^{\omega_1})^+)$*-extendible.*

THEOREM 19. *If it is consistent there is a weakly compact cardinal, then it is consistent that CH holds and* $[2^{\omega_1}]^{<2^{\omega_1}}$ *is* $(2^{\omega_1}, 2^{\omega_1})$*-extendible.*

PROOF. Let $P_\xi = \cup\{{}^\Gamma 2; \Gamma \in [\xi]^{\leqq \aleph_0}\}$. The ordering on P_ξ is reverse inclusion. If $V \models$"$2^{\aleph_0} = \aleph_1$" and G is P_κ-generic over V, where κ is weakly compact, then $V[G] \models$"$2^{\aleph_1} = \kappa$ and $2^{\aleph_0} = \aleph_1$". It suffices to show that $[2^{\omega_1}]^{<2^{\aleph_1}}$ is $(2^{\aleph_1}, 2^{\aleph_1})$-extendible in $V[G]$. Now suppose that $V[G] \models$"$\{A_\alpha; \alpha \in \kappa\}$ witnesses that $[\kappa]^{<\kappa}$ is not (κ, κ)-extendible". It will now be shown that there is $C \in [\kappa]^\kappa$ in $V[G]$ such that C is closed under increasing sequences of uncountable cofinality and if $\alpha \in C$ and $\mathrm{cf}(\alpha) > \omega$ then $\{A_\beta \cap \alpha; \beta \in \alpha\}$ has the following property:

there is $B \subseteq \alpha$ such that for each $\Gamma \in [\alpha]^{\aleph_0}$,

(1) $$(\cup\{\alpha - A_\gamma; \gamma \in \Gamma - B\}) \cup (\cup\{A_\gamma \cap \alpha; \gamma \in \Gamma \cap B\}) \not\subseteq \beta \text{ for any } \alpha - \beta.$$

To see this define $h : \kappa \times [\kappa]^{\aleph_0} \times [\kappa]^{\aleph_0} \to \kappa$ such that $h(\zeta, B^0, B^1)$ is a member of $(\kappa - \zeta) - ((\cup\{\alpha - A_\beta; \beta \in B^0\}) \cup (\cup\{\alpha \cap A_\beta; \beta \in B^1\}))$ whenever

$$\kappa - \zeta \not\subseteq (\cup\{\alpha - A_\beta; \beta \in B^0\}) \cup (\cup\{\alpha \cap A_\beta; \beta \in B^1\}).$$

Let $C = \{\alpha \in \kappa; \mathrm{cf}(\alpha) > \omega$ and $h''(\alpha \times [\alpha]^{\aleph_0} \times [\alpha]^{\aleph_0}) \subseteq \alpha\}$. Then if $\alpha \in C$ define $B = \{\xi \in \alpha; \alpha \notin A_\xi\}$. To see that B has the desired property suppose that $\Gamma \in [\alpha]^{\aleph_0}$. Let $\beta \in \alpha$. Then, by the definition of B,

$$\beta \notin (\cup\{\alpha - A_\gamma; \gamma \in \Gamma - B\}) \cup (\cup\{\alpha \cap A_\gamma; \gamma \in \Gamma \cap B\}).$$

Then, by the definition of C,

$$h(\beta, \Gamma - B, \Gamma \cap B) \in \alpha$$

and $h(\beta, \Gamma - B, \Gamma \cap B)$ does not belong to

$$(\cup\{\alpha - A_\gamma ; \gamma \in \Gamma - B\}) \cup (\cup\{\alpha \cap A_\gamma ; \gamma \in \Gamma \cap B\}) \cup \beta$$

which shows that (1) holds.

Since P_κ satisfies the $\aleph_2$-chain condition, C contains an unbounded set in the ground model which is closed under increasing uncountable sequences. Hence, without loss of generality, it can be assumed that $C \in V$. Let $\underline{A_\beta} \subseteq \kappa \times [P_\kappa]^{<\aleph_2}$ be a name for A_β. Because of the $\aleph_2$-chain condition it can be assumed that $\underline{A_\beta} \subseteq V_\kappa$. Then

(2)
$$(V_\kappa, P_\kappa, \{\underline{A_\beta} ; \beta \in \kappa\}) \models (\forall X \subseteq \kappa \times [P_\kappa]^{<\aleph_2})$$
$$(\forall Y \subseteq \kappa \times [P_\kappa]^{<\aleph_2})[(\forall p \in P_\kappa)(p \Vdash_{P_\kappa} \text{“} X \in [\kappa]^\kappa,\ Y \in [\kappa]^\kappa \text{ and } X \cap Y = 0\text{”})$$
$$\text{implies } (\exists \Gamma \in [\kappa]^{\aleph_0})(\exists q \in P_\kappa)(\exists \gamma \in \kappa)(\forall \eta \in \kappa - \gamma)(\exists \beta \in \Gamma)(q \leqq p \text{ and}$$
$$q \Vdash_{P_\kappa} \text{“}\Gamma \subseteq X \cup Y \text{ and } (\eta \in \underline{A_\beta} \text{ if } \beta \in X) \text{ and } (\eta \notin \underline{A_\beta} \text{ if } \beta \in Y)\text{”}))].$$

It is easy to check that the expression in (2) involving $\Vdash$ can be replaced by first order expressions in the language of the model

$$(V_\kappa, P_\kappa, \{\underline{A_\beta} ; \beta \in \kappa\}).$$

Hence the entire expression in square brackets is first order with respect to this language. Hence, since κ satisfies the Π_1^1-reflection property, it is possible to find $\lambda \in C$ such that

$$(V_\lambda, P_\lambda, \{\underline{A_\beta} \cap (\lambda \times [P_\lambda]^{<\aleph_2}); \beta \in \lambda\}) \models (\forall X \subseteq \lambda \times [P_\lambda]^{<\aleph_2})$$
$$(\forall Y \subseteq \lambda \times [P_\lambda]^{<\aleph_2})[(\forall p \in P_\lambda)(p \Vdash_{P_\lambda} \text{“} X \in [\lambda]^\lambda,\ Y \in [\lambda]^\lambda \text{ and } X \cap Y = 0\text{”})$$
$$\text{implies } (\exists \Gamma \in [\lambda]^{\aleph_0})(\exists q \in P_\lambda)(\exists \gamma \in \lambda)(\forall \eta \in \lambda - \gamma)(\exists \beta \in \Gamma)(q \leqq p \text{ and}$$
$$q \Vdash_{P_\lambda} \text{“}\Gamma \subseteq X \cup Y \text{ and } (\eta \in \underline{A_\beta} \cap (\lambda \times [P_\lambda]^{<\aleph_2}) \text{ if } \beta \in X) \text{ and}$$
$$(\eta \notin \underline{A_\beta} \cap (\lambda \times [P_\lambda]^{<\aleph_2}) \text{ if } \beta \in Y)\text{”}))].$$

In particular

(3) $V[G \cap P_\lambda] \models$“$\{A_\beta \cap \lambda ; \beta \in \lambda\}$ witnesses that $[\lambda]^{<\lambda}$ is not (λ, λ)-extendible”.

But by (1)

$$V[G\cap P_\kappa]\models\text{“}(\exists B\subseteq\lambda)(\forall\Gamma\in[\lambda]^{\aleph_0})(\forall\beta\in\lambda)((\cup\{\lambda-A_\gamma;\gamma\in\Gamma-B\})\cup(\cup\{\lambda\cap A_\gamma;\gamma\in B\cap\Gamma\}))\not\supseteq\lambda-\beta\text{”}.$$

But since P_κ is countably closed and satisfies the $\aleph_2$-chain condition it follows that

$$(4)\qquad\begin{cases} V[G\cap P_\lambda]\models\text{“there is a } P_\lambda\text{-name such that}\\ 1\Vdash_{P_\lambda}\text{“}(\forall\Gamma\in[\lambda]^{\aleph_0})(\forall\beta\in\lambda)((\cup\{\lambda-A_\gamma;\gamma\in\Gamma-B\})\\ \cup(\cup\{\lambda\cap A_\gamma;\gamma\in B\cap\Gamma\})\not\supseteq\lambda-\beta)\text{”}.\end{cases}$$

But $V[G\cap P_\lambda]\models$“$2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\lambda$”. Hence

$$V[G\cap P_\lambda]\models\text{“}P_\lambda=\{Q_\xi;\xi\in\omega_1\}\text{ where each } Q_\xi \text{ is } \sigma\text{-directed”}.$$

Therefore there is, in $V[G\cap P_\lambda]$, $B^*\in[\lambda]$ and $\eta\in\omega_1$ such that

$$V[G\cap P_\lambda]\models\text{“}(\forall\beta\in B^*)(\exists p_\beta\in Q_\eta)(p_\beta\Vdash_{P_\lambda}\text{“}\beta\in B\text{” or } p_\beta\Vdash_{P_\lambda}\text{“}\beta\notin B\text{”})\text{”}.$$

Now let $X=\{\beta\in B^*;p_\beta\Vdash_{P_\lambda}\text{“}\beta\in B\text{”}\}$ and $Y=\{\beta\in B^*;p_\beta\Vdash_{P_\lambda}\text{“}\beta\notin B\text{”}\}$. Clearly $\{X,Y\}\subseteq V[G\cap P_\lambda]$ and $X\cap Y=0$. But by (3) there is $\Gamma\in[B^*]^{\aleph_0}$ and $\xi\in\lambda$ such that

$$(\cup\{\lambda-A_\gamma;\gamma\in\Gamma\cap X\})\cup(\cup\{\lambda\cap A_\gamma;\gamma\in\Gamma\cap Y\})\supseteq\lambda\setminus\xi.$$

Choosing $q\in P_\lambda$ such that $(\forall\lambda\in\Gamma)(q\supseteq p_\gamma)$ yields a contradiction to (4).

QUESTION 9. Does the consistency of the $(2^{\omega_1},2^{\omega_1})$-extendibility of $[2^{\omega_1}]^{<2^{\omega_1}}$ imply the consistency of the existence of an inaccessible cardinal?

Todorčević has subsequently proved in [19] that Theorem 19 can be obtained by adding weakly compact many Sacks subsets of ω_1 (see comment at the end of claim 3 in section 2 of his paper).

Todorčević has also obtained (see property (3) of section 2 of [19]) by adding weakly compact many Sacks reals:

THEOREM 20 (Todorčević). *If it is consistent that there is a weakly compact cardinal, then it is consistent with $2^\omega=2^{\omega_1}$ that $[2^{\omega_1}]^{<2^{\omega_1}}$ is $(2^{\omega_1},2^{\omega_1})$-extendible.*

PROOF. (Let $\lambda=2$ where $\kappa=2^\omega$ in aforementioned property (3).)

COROLLARY. *If it is consistent that there is a weakly compact cardinal, then it is consistent that every 2^{ω_1}-completable ideal is $(2^{\omega_1},2^{\omega_1})$-extendible.*

PROOF. Let I be an ideal and let $\{A_\xi : \xi \in 2^{\omega_1}\}$ witness that I is not $(2^{\omega_1}, 2^{\omega_1})$-extendible. Whenever X and Y are disjoint countable subsets of 2^{ω_1}, let $B_X^Y \in I$ be such that

$$B_X^Y \cup \bigcup \{A_\xi : \xi \in X\} \cup \bigcup \{(\bigcup I) - A_\xi : \xi \in Y\} = \bigcup I$$

if possible. Then $\{A_\xi : \xi \in 2^{\omega_1}\}$ witnesses that the ideal generated by the B_Y^X is not $(2^{\omega_1}, 2^{\omega_1})$-extendible. We assume therefore that I has 2^{ω_1}-many generators. There is a set B of positive measure which is such that $|B \cap A| < 2^{\omega_1}$ for each $A \in I$. An application of Theorem 19 to the restriction of I to B completes the proof.

LEMMA 16. *If each μ-completable ideal with κ^ω-generators is (κ, λ)-extendible, then every μ-completable ideal is (κ, λ)-extendible.*

THEOREM 21 (Kunen). *It is consistent with any cardinal arithmetic that $[2^{\omega_1}]^{<2^{\omega_1}}$ is (ω, ω)-extendible.*

PROOF. (See p. 289 A10 of [12]). It is consistent with any cardinal arithmetic that there is an ultrafilter U on ω of character ω_1. Let $\{A_\gamma : \gamma \in \omega_1\}$ be a base for U. Let $\{B_n : n \in \omega\}$ be a family of subsets of 2^{ω_1}. Let $\{C_\alpha : \alpha \in 2^{\omega_1}\}$ be a family of subsets of ω defined by $n \in C_\alpha$ iff $\alpha \in B_n$. Assume that there is $X \in [2^{\omega_1}]^{2^{\omega_1}}$ such that $\alpha \in X$ implies $C_\alpha \in U$. $\mathrm{cf}(2^{\omega_1}) > \omega_1$ implies that, without loss of generality, there is $\gamma \in \omega_1$ such that $\alpha \in X$ implies $C_\alpha \supset A_\gamma$. For each $n \in A_\gamma$, $B_n \supset X$ as required.

The authors thank B. Veličković, A. Miller, S. Todorčević and the referee for enabling us to renumber many questions as theorems.

We conclude with the general question of "third order" extendibility: When does a σ-ideal I have the property that any family of κ sets has a subfamily of λ-sets, any μ-many of which can be measured by a σ-ideal which extends I?

REFERENCES

1. J. E. Baumgartner, *Iterated forcing*, in *Surveys in Set Theory* (A. D. Mathias, ed.), London Math. Soc. Lecture Notes #87, Cambridge Univ. Press, 1983, pp. 1–59.

2. J. E. Baumgartner and A. Hajnal, *A proof (involving Martin's axiom) of a partition relation*, Fund. Math. **78** (1973), 193–203.

3. J. E. Baumgartner and R. Laver, *Iterated perfect-set forcing*, Ann. Math. Logic **17** (1979), 271–283.

4. W. Boos, *Infinitary compactness without strong inaccessibility*, J. Symb. Logic **41** (1976), 33–38.

5. W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974.

6. A. J. Dodd, *The Core Model*, London Math Soc. Lecture Notes, Series No. 61, Cambridge Univ. Press, 1982.

7. P. Erdös, A. Hajnal and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196.

8. A. Hajnal, *Combinatorics*, Cambridge Summer School Notes 1978, preprint.

9. A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, in *Higher Set Theory*, Lecture Notes in Mathematics **669**, Springer-Verlag, Berlin, 1977.

10. H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math. **53** (1965), 225–308.

11. K. Kunen, *Saturated ideals*, J. Symb. Logic **43** (1978), 65–76.

12. K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1983.

13. R. Laver, *A saturation property on ideals*, Compositio Math. **36** (1978), 223–242.

14. R. Laver, *An* $(\aleph_2,\aleph_0,\aleph_0)$*-saturated ideal on* ω_1, Logic Colloquium '80, North-Holland, Amsterdam, 1982, pp. 173–180.

15. R. Laver, *Making the supercompactness of* κ *indestructible under* κ*-directed closed forcing*, to appear.

16. T. K. Menas, *Consistency results concerning supercompactness*, Trans. Amer. Math. Soc. **223** (1976), 61–91.

17. K. L. Prikry, *On a problem of Erdös, Hajnal and Rado*, Discrete Math. **2** (1972), 51–59.

18. S. Shelah, *Proper Forcing*, Springer Lecture Notes # 940, Springer-Verlag, New York, 1982.

19. S. Todorčević, *Reals and positive partition relations*, to appear.

20. N. H. Williams, *Combinatorial Set Theory*, Studies in Logic and Foundations of Mathematics 91, North-Holland, Amsterdam, 1977.